



SB-3573

M. A. / M. Sc. (Part - II) Examination

March / April - 2011

Mathematics : Paper - 501

(Advanced Functional Analysis)

(Old Course)

Time : 3 Hours]

[Total Marks : 70

Instructions :

(1)

નીચે દર્શાવેલ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
 Fillup strictly the details of signs on your answer book.

Name of the Examination :

Name of the Subject :

Subject Code No. : Section No. (1, 2,.....) :

Seat No. :

Student's Signature

- (2) Answer all questions.
 (3) Each question carries 14 marks.
 (4) Figures to the right indicate marks of the question.
 (5) Follow usual notations.

- 1 (a) Let X and Y be Banach spaces and $T : D(T) \rightarrow Y$ a closed linear operator, where $D(T) \subset X$. Then prove that if $D(T)$ is closed in X , the operator T is bounded. **6**
- (b) If the dual space X' of a normed space X is separable, then X itself is separable. **4**
- (c) Show that $p(x) = \overline{\lim}_{n \rightarrow \infty} \xi_n$, where $x = (\xi_n) \in l^\infty$ and ξ_n real, defines a sublinear functional on l^∞ . **4**

OR

- 1 (a) Let X and Y be normed spaces and $T : X \rightarrow Y$ be a bounded linear operator. Then prove that adjoint operator T^\times is linear, bounded and $\|T^\times\| = \|T\|$. **6**
- (b) Show that in finite dimensional normed spaces the distinction between weak convergence and strong convergence disappears. **4**

- (c) Let X and Y be normed spaces, $T \in B(X, Y)$ and (x_n) a sequence in X . Show that 4

$$x_n \xrightarrow{w} x_0 \Rightarrow Tx_n \xrightarrow{w} Tx_0.$$

- 2 (a) Let X be a complex Banach space, $T \in B(X, X)$ 10

and $P(\lambda) = \sum_{i=0}^n \alpha_{n-i} \lambda^{n-i}, \alpha_n \neq 0$. Then show that

$$\sigma(P(T)) = P(\sigma(T)).$$

- (b) If X is a complex Banach space, $S, T \in B(X, X)$ 4
and $ST = TS$, then prove that

$$r_\sigma(ST) \leq r_\sigma(S)r_\sigma(T)$$

Can we drop $ST = TS$? Why ?

OR

- 2 (a) Prove that all the matrices representing a given linear operator $T: X \rightarrow X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values. 6

- (b) Prove that eigenvectors x_1, x_2, \dots, x_n corresponding to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of a linear operator T on a vector space X constitute a linearly independent set. 4

- (c) Let $X = C[0, \pi]$ and define $T: D(T) \rightarrow X$ by $x \mapsto x''$, 4
where

$$D(T) = \{x \in X \mid x', x'' \in X, x(0) = x(\pi) = 0\}.$$

Show that $\sigma(T)$ is not compact.

- 3 (a) Let X and Y be normed spaces and $T: X \rightarrow Y$ a compact linear operator. Suppose (x_n) in X is 6

weakly convergent. Then prove that (Tx_n) is strongly convergent in Y with limit $y = Tx$.

- (b) Show that the range of a compact linear operator $T: X \rightarrow Y$ is separable, where X and Y are normed spaces. 4

- (c) Prove that $T: l^2 \rightarrow l^2$ defined by $Tx = y = (\eta_j)$, 4
 where $n_j = \xi_j / j$ is compact.

OR

- 3 (a) Show that a compact linear operator $T: X \rightarrow Y$ form 6
 a normed space X into a Banach space Y has a
 compact linear extension $\tilde{T}: \hat{X} \rightarrow Y$, where \hat{X} is the
 completion of X .
- (b) Let $T: X \rightarrow X$ be a compact linear operator on a 4
 normed space X . Then prove that for every $\lambda \neq 0$
 the null space $N(T_\lambda)$ of $T_\lambda - \lambda I$ is finite
 dimensional. Hence show that $\dim N(T_\lambda^n) < \infty$.
- (c) Let H be a Hilbert space, $T: H \rightarrow H$ a bounded 4
 linear operator and T^* , the Hilbert adjoint
 operator of T . Show that T is compact if and
 only if T^*T is compact.
- 4 (a) Let $T: H \rightarrow H$ be a bounded self-adjoint linear 6
 operator on a complex Hilbert space H . Show
 that $\lambda \in \rho(T) \Leftrightarrow$ there exists $c > 0$ such that for
 every $x \in H, \|T_\lambda x\| \geq c\|x\|$.
- (b) Let $H \neq \{0\}$ and $T: H \rightarrow H$ be a bounded 4
 self-adjoint linear operator on a complex Hilbert
 space H . Show that $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ is spectral
 value of T .
- (c) Let T be any bounded linear operator on a 4
 complex Hilbert space. Show that the inverse of
 $I + T^*T$ exists.

OR

- 4 (a) Let P_1 and P_2 be projections on a Hilbert space H . 6
 Then show that if $P = P_1 + P_2$ is a projections on
 $H \Leftrightarrow Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

- (b) Show that the residual spectrum $\sigma_r(T)$ of a bounded self-adjoint linear operator $T: H \rightarrow H$ on a complex Hilbert space H is empty. 4
- (c) Show that if $T \geq 0$, then $(I + T)^{-1}$ exists. 4
- 5 (a) Let H be a complex Hilbert space and $T: D(T) \rightarrow H$ be a self-adjoint linear operator with $D(T)$ dense in H . Show that the spectrum $\sigma(T)$ is real. 8
- (b) Let H be a complex Hilbert space and $T: D(T) \rightarrow H$ linear and densely defined in H . Then show that T is symmetric $\Leftrightarrow \langle Tx, x \rangle$ is real for all $x \in D(T)$. 6

OR

- 5 (a) Prove that if a linear operator T is defined on all of complex Hilbert space H and satisfies $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$, then T is bounded. 6
- (b) Let $S: D(T) \rightarrow H$ and $T: D(T) \rightarrow H$ be linear operators which are densely defined in a complex Hilbert space H . Then show that $S \subset T \Rightarrow T^* \subset S^*$. 4
- (c) Show that if T is symmetric, then T^{**} is symmetric. 4